

Link Theory – from Computation to Quantum Physics

or

*How to build a Universe using only things
found lying around the math department*

Part I: the Basics

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Abstract: Part I of this paper presents the basic ideas and mechanics of Link Theory, along with several examples from the realm of digital circuits, computation, and physics. In Part II, we show how the same principles can be used to derive the core laws of Quantum Physics, and how these laws are properties of the mathematics, not miracles, and not in any way derived from properties of physical matter.

0. INTRODUCTION

0.1 What is Link Theory?

Link Theory is essentially a calculus of composite abstract relations. As such, it is *not* itself a theory of computation or physics (or anything else), but a *tool* for creating and analyzing structures which represent digital circuits, physical situations including quantum physics experiments, and much more. Several examples are given below. Link Theory was developed primarily by Tom Etter beginning in 1996, but it has origins in much earlier work as well.

A *relation* is a set of mutually consistent *possibilities* among several variables. This set is in principle finite. Link Theory shows how to represent relations in tabular form, and how to compose relations into larger relations. A relation may also be viewed as a *constraint* on the set of all possible mutual values of several variables.

By *abstract* relation, we mean something which represents structural properties of a relation that don't depend on the kinds of things which are thus related. This

means, for instance, that the structure of the Boolean relation ($A = B \text{ AND } C$), as given by a table containing values 0 and 1, could just as well be given by a table with values “true” and “false”, or “good” and “bad”, or “Mars” and “Venus”, or 0 and 1 reversed. The abstract AND relation is that which all such tables have in common.

1. LINK THEORY – THE BASICS

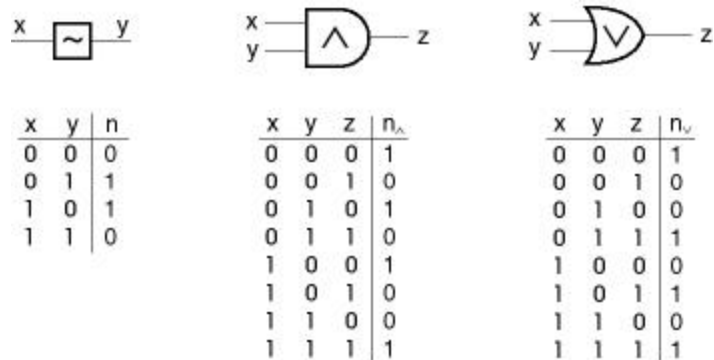
In this section, we describe the mechanics of Link Theory and give several examples of its use in representing digital circuits, solving algebraic equations, and studying computations in a computer operating on quantum mechanical principles.

Link Theory is a very simple calculus of composite relations in tabular form. It is closely related to tensor algebra of quantum theory, but can be used in a simpler form to represent digital circuits and other forms of computation. The great strength of Link Theory is its straightforward representation of *what is so* – or more to the point, *what is possible*.

1.1. Variables, domains, cases, case counts

To get started, we take the simple binary domain of digital logic. Each *variable* or signal can take on either of two *values*, which we designate as **0** and **1**. The value of a variable is seldom in any sense actual or definite in this calculus, but rather *the values represent possibilities*. It is from a simple enumeration of possibilities which we will build our universe.

Here is a set of common logic gates, each represented by a table consisting of cases, or rows, which enumerate all possible combinations of values among the variables indicated. Note that this representation is entirely symmetric, and no distinction is made between inputs and outputs. Unlike real electronic gates, no direction of action or causal relationship is implied or enforced here.



Each table also includes an extra column labelled **n**, called the *case count* for that row. The case count may be thought of as the number of ways in which that combination of values can occur. In our initial examples, a case count will usually be 0 (impossible) or 1 (possible). A short inspection of the tables above will show that 1s in the case count columns correspond to the usual functions of these familiar gates. (It is important to distinguish the logical *values* designated **0** and **1** from the case counts 0 and 1, which are *numbers*. We could have used another nomenclature here, but both uses are standard and familiar.)

In general, case counts can be greater than 1, indicating hidden variables, and *less than 0* as required to represent many quantum situations. The meaning of a *negative possibility* is indeed interesting to contemplate, but is mostly beyond the scope of this paper.

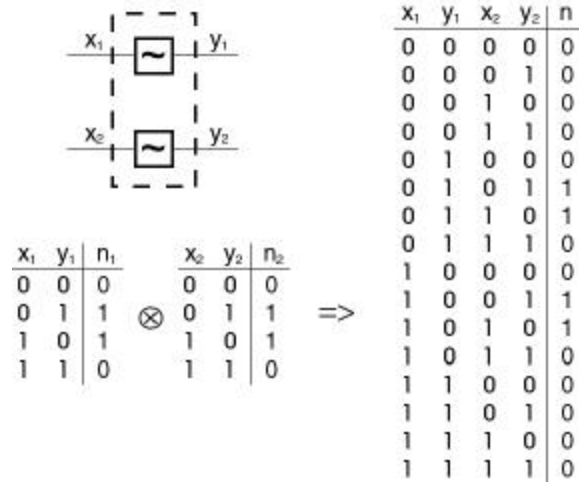
1.2. Relations

A Link Table may also be regarded as a *relation* or *constraint* on the joint values of its variables. Those cases which have a case count > 0 are permitted, that is *valid* under the constraint, those with case count $= 0$ are disallowed or impossible. A relation is inherently symmetrical, having no “input” or “output”, unlike our usual conception of digital circuits. Relations are in general not functions in the mathematical sense, but all functions are relations. The bi- (multi-) directional character of relations can be used to great advantage in synthesizing and analyzing circuits.

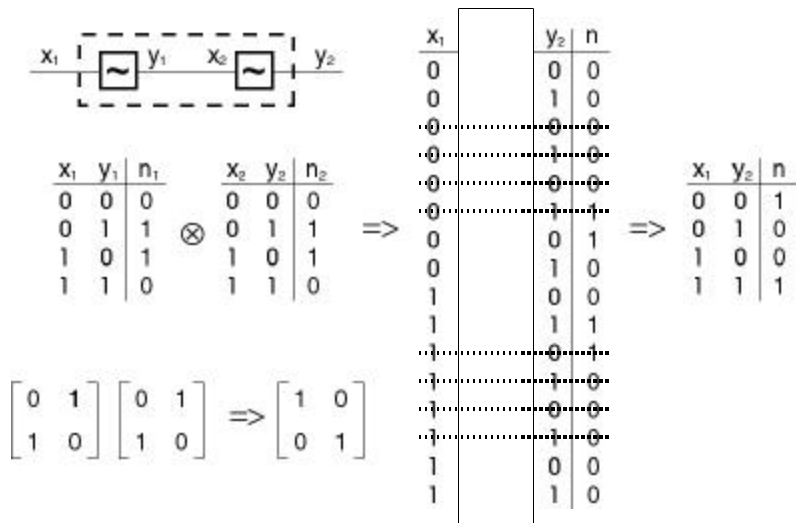
1.3. Composing relations

Composing Link Tables is a simple matter of combinatorics -- accounting for all the possibilities. A composite *linked* relation or constraint is a *limitation* on the set of all the joint possibilities for the combined set of variables.

For example, if we consider two NOT gates taken together (but not connected), there are a total of four binary variables, and thus 16 total cases in their Cartesian product. But since both NOT constraints must be satisfied, only 4 cases are actually valid. *The case count in the composite table is derived by simply multiplying the case counts of the components for each case.*



The diagram below again shows two NOTs, but with the addition of a connection, or *link*, between variables y_1 and x_2 . In the composite table, these two variables must therefore have the same value in all cases, so 8 cases where this is not so have been crossed out. Note that only two of these were valid in the first place.

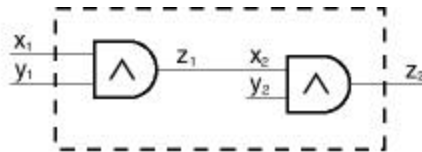


Furthermore, we don't care at the moment what the values of y_1 and x_2 are, so we *hide* them, that is, we do not connect them outside the new composite circuit (dashed box). Variables x_1 and y_2 remain, and are visible and of interest. To get the overall result, we can simply erase the column associated with the hidden variable (the composite y_1-x_2), and *sum the case counts of like cases*. Four cases of the two variables remain, each with two instances in the composite table.

Summing, we get four cases, only two of which are possible. Apparently, linking two NOTs in this way yields the *identity relation*, since only the cases **0-0** and **1-1** have case counts = 1.

Note that we have included at the bottom of the figure an equivalent matrix multiplication, where the case counts are taken as matrix coefficients, each indexed by the corresponding logic values. Later we show the equivalence of Linking to matrix (more generally, tensor) multiplication.

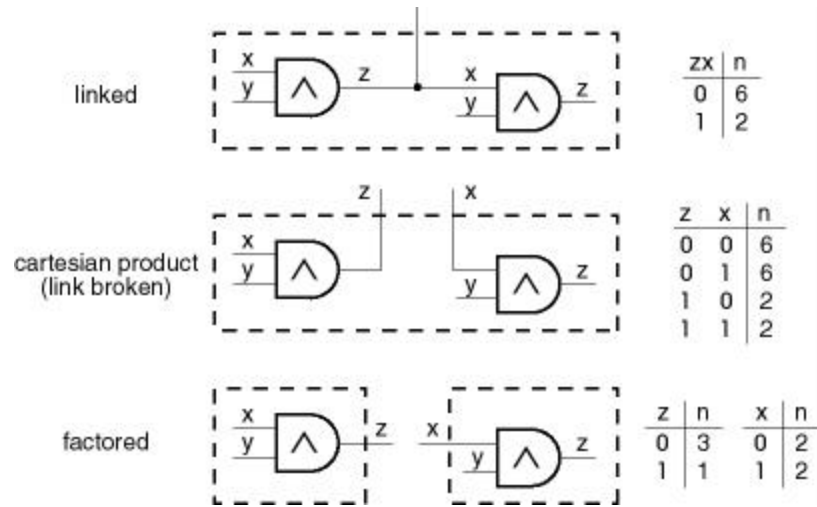
Variables may be hidden or disregarded, as shown above, for various reasons. In the example below, we want to link the output of the first gate (z_1) to one of the inputs of the second gate (x_2), and then determine the relation which results between inputs x_1 and y_1 and the output z_2 . We choose to hide or ignore both the linked variable z_1 - x_2 and also the other input y_2 . (An engineer would call y_2 a "don't care".) To do this, we link variables z_1 and x_2 in the AND gate tables as above, then hide (erase) the columns for z_1 - x_2 and y_2 , and combine like cases by summing their case counts.



x_1	y_1	z_1	n_x		x_2	y_2	z_2	n_x		x_1	y_1	z_2	n
0	0	0	1		0	0	0	1	\Rightarrow	0	0	0	2
0	0	1	0		0	0	1	0		0	0	1	0
0	1	0	1	\oplus	0	1	0	1		0	1	0	2
0	1	1	0		0	1	1	0		0	1	1	0
1	0	0	1		1	0	0	1		1	0	0	2
1	0	1	0		1	0	1	0		1	0	1	0
1	1	0	0		1	1	0	0		1	1	0	1
1	1	1	1		1	1	1	1		1	1	1	1

1.4. Breaking a Link

Links may also be *broken*, and components which are thus separated are called *factored*. Below we show two AND gates linked much as before, but this time with only the linked variable z - x visible. (We have hidden all the others for this example. The reader may verify that case counts of 6 and 2 are the result.) We then break the link z - x , thus separating the variables z and x again. What remains is the cartesian product of the two tables taken together, with only z and x visible. (In physics terms, this is the density matrix relating the two variables) Finally, we separate the two components into two tables whose product is the composite table. This step is equivalent to factoring a matrix, as will be seen in Part II.

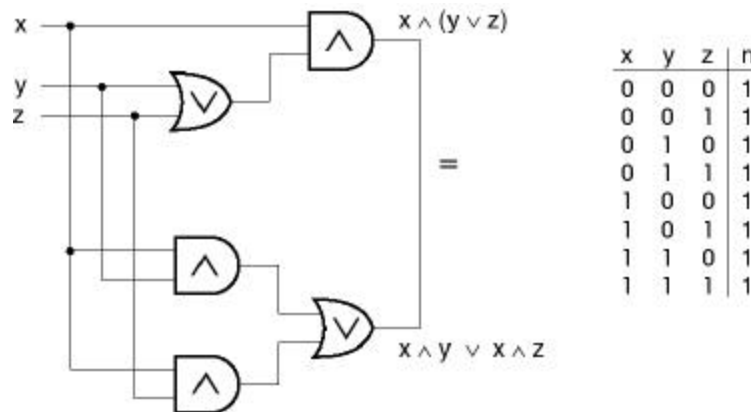


2. EXAMPLES AND EXTENSIONS

2.1 Example: Distributive Law

Below is somewhat larger example, again in the logic domain. By combining instances of the tables given above, and linking variables as previously described, we arrive at a simple “proof” of a Distributive Law of logic (AND distributes over OR). To represent the claimed equality, we simply connect (link) the two expressions together – something ordinary circuits (and most circuit tools) would not allow.

For every combination of values for x , y , and z , the composite relation is valid, as shown by the resulting case counts. If the circuit did not yield a tautology, the counts would indicate exactly those cases where the theorem failed.



2.2. Example: Arithmetic

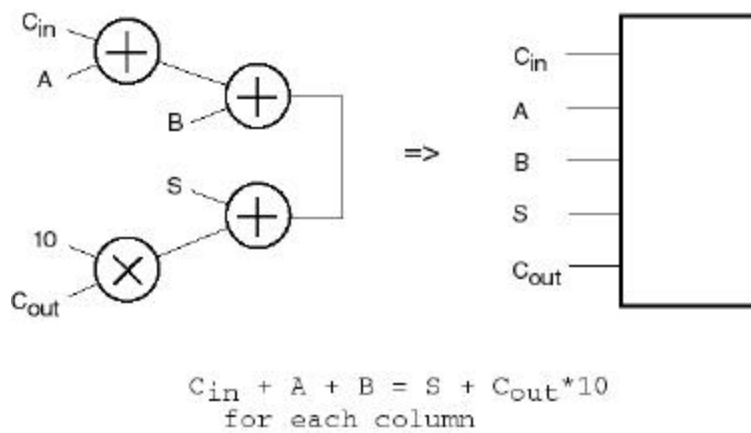
Link Theory has been successfully applied not only to logic domain problems, but also to numerical problems such as cryptarithms (shown here), finite-domain algebraic problems such as Diophantine (integer-solution) equations, and more. Here is a cryptarithm example usually considered in the domain of Artificial Intelligence. The problem is to find an assignment of single digits 0-9 to the letters such that the sum is satisfied. Each letter must correspond to a unique digit.

$$\begin{array}{r}
 \text{SEND} \\
 +\text{MORE} \\
 \hline
 \text{MONEY}
 \end{array}
 \Rightarrow
 \begin{array}{r}
 D + E = Y + C_1 * 10 \\
 C_1 + N + R = E + C_2 * 10 \\
 C_2 + E + O = N + C_3 * 10 \\
 C_3 + S + M = O + C_4 * 10 \\
 C_4 = M
 \end{array}
 \left| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right.$$

with $S, E, N, D, M, O, R, Y \in \{0-9\}$
uniquely, and $C_i \in \{0,1\}$

Also shown are the simultaneous equations implied by this sum, one equation per column. C_1 represents the carry out of the right-most column, and so on. We also assume that positional numbers do not begin with a leading 0, so M must equal 1.

To solve this problem using Link Theory, we first create a Link Diagram as shown below to represent each column constraint. Tables are used to represent the addition operator (only up to a sum of 19) and multiplication by 10. Reduction of this Link Diagram results in a single 5-column table, as shown.



By linking together 4 such composite tables according to the problem (linking **E** in column 1 to **E** in columns 2 and 3, for example), and applying the uniqueness constraints ($S \neq E$, $S \neq N$, $S \neq D$, ..., $E \neq N$, $E \neq D$, ..., $R \neq Y$) similarly, and reducing the overall composite Link Table, we are left with a table containing only one case, the solution:

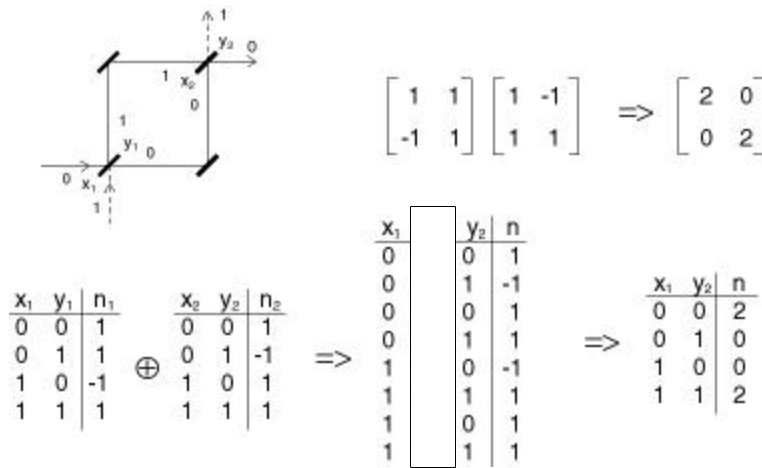
$$\begin{array}{r}
 \text{SEND} \\
 +\text{MORE} \\
 \hline
 \text{MONEY}
 \end{array}
 \Rightarrow
 \begin{array}{r}
 9567 \\
 +1085 \\
 \hline
 10652
 \end{array}$$

2.3. Example: Quantum interference

In this example, we introduce a “gate” whose table includes a *negative case count*. (The deeper meaning of case counts < 0 is beyond the scope of this paper.) The diagram below represents an interferometer consisting of four mirrors -- two half-silvered and two fully-silvered. Four variables will be used to describe the possible paths of a photon through the apparatus.

A photon enters from the left (signified by variable $x_1 = 0$) or from below ($x_1 = 1$) and impinges on the first half-silvered mirror. Variable y_1 indicates the direction in which the photon leaves (upward, $y_1 = 1$, or to the right, $y_1 = 0$). The photon then bounces from a fully-silvered mirror and arrives at the second half-silvered mirror from the left ($x_2 = 1$) or from below ($x_2 = 0$), and exits upwards ($y_2 = 1$) or to the right ($y_2 = 0$). (For this example, we have simplified the actual physical situation and neglected complications introduced by the fully-reflecting mirrors.)

Using negative case counts, we can model a crucial aspect of the situation which actually occurs in nature. Tables for the two half-silvered mirrors are shown, and their the composite table, along with the corresponding matrix representation. From the matrices, it is easy to see that the half-silvered mirror acts as a 45° rotation on the photon state. In quantum physical terms, we have applied a *unitary transform* to the photon, and created a *superposition* of the two paths through the apparatus.



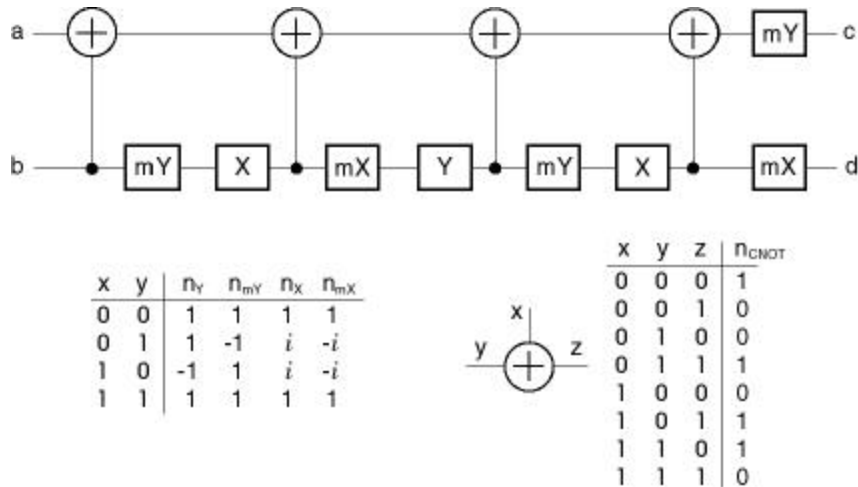
By hiding the linked variables y_1 and x_2 , we are left with two cases for each of the four values of x_1 and y_2 , similarly to the previous example. However, in two cases the negative values cancel with the positive values, and the result is 0, or impossibility -- in physical terms, *interference*. Thus from the composite table, we can see that any photon entering from the left *must* exit from the right, and a photon entering from below will necessarily exit upwards. (In Part II, we explore the connection to quantum physics in much greater detail.)

2.4. Quantum computation

Recently, a number of research groups have been experimenting with computational mechanisms which operate in the quantum realm in hopes of building a *quantum computer*. A variable in a quantum computer can in principle represent a superposition of several states at once.

The following quantum computer circuit (or algorithm, equivalently) was suggested to us by Isaac Chuang of IBM Almaden Research Center. Using Link Tables, we calculated the composite transformation as described above, and achieved the identical result that standard quantum theory does. The reader may verify that compositing the given tables as shown in the diagram does give the result shown on the right.

Note that the imaginary case counts of $+i$ and $-i$ required by the given components can be accommodated by introducing two new variables linked in common among all elements, thus quadrupling the number of cases. This is equivalent to substituting a 2×2 spinor with real values for the imaginary matrix coefficients, as is sometimes done in quantum theory. This shows that imaginary case counts are not needed as a further extension to the theory, and in fact may be preventing a deeper understanding of the underlying structure of nature.



3. THE BRIDGE TO PHYSICS NOTATION

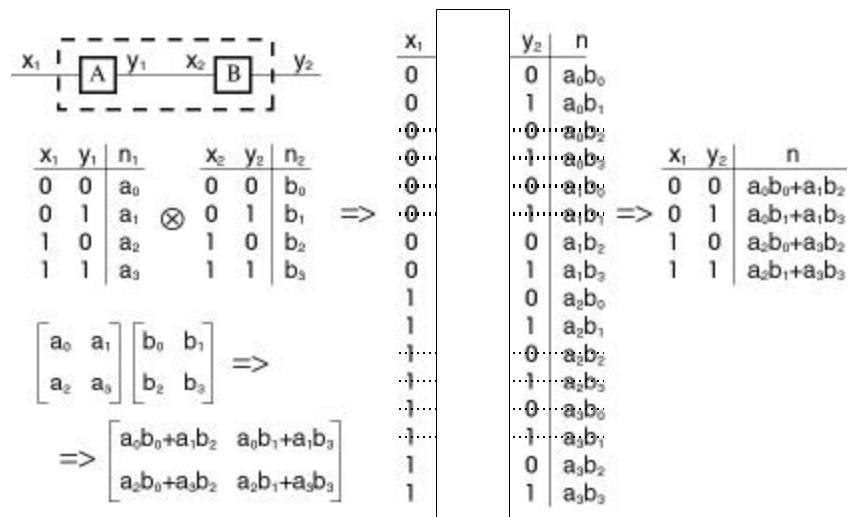
In a physicist's terms, the general binary pure state (in Dirac notation) $a|0\rangle + b|1\rangle$, corresponds to the Link Table

x	n
0	a
1	b

where a and b are scalars, and $|0\rangle$ and $|1\rangle$ represent the states corresponding to the logical values of 0 and 1 .

3.1. Equivalence to matrices/tensors

Returning to our simple two-component two-variable situation, if we substitute variables for the usual numerical case counts, and perform linking and reduction as before, it is easy to see that the result is the same as multiplying the corresponding matrices.



More generally, Link Tables can represent higher-dimensional matrices (tensors) and operations upon them. The advantage of Link Theory is a much simpler, more intuitive notation, and a more direct connection to the underlying combinatorics of the physical situation. With case counts > 0 , we can represent logic circuits, sets, etc., while case counts < 0 are necessary to fully represent quantum phenomena.

4. SUMMARY AND IMPLICATIONS

We have shown how Link Theory, a simple accounting of possibilities, can be used to represent logic and idealized digital circuits, arithmetic problems, quantum experiments and much more. We also showed the equivalence of Link Theory to matrices and tensor algebra.

Link Theory is in fact a *general theory of structure*. Using Link Theory, we can show the equivalence and correspondence of theories in previously completely disparate realms, and *thus bring the theorems and the power of each to the others*.

In Part II, we show how the core laws of quantum mechanics can be derived from simple, entirely mathematical considerations.

5. REFERENCES AND RELATED WORK

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